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LENS SPACE SURGERIES ALONG TWO COMPONENT LINKS AND REIDEMEISTER-TURAEV TORSION

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1. INTRODUCTION

This article is a short survey of the core part of the authors' joint work "Lens space surgeries along certain 2-component links, Park's rational blow down, and Reidemeister–Turaev torsion" [KTYY, preprint]. In [KTYY], certain two families of 2-component links, denoted by $A_{m,n}$ and $B_{p,q}$ are focused, and the main result is the decision of the coefficient(s) of the knotted component yielding a lens space by Dehn surgery. The links are related to rational homology 4-ball used in J. Park's (generalized) rational blow down in 4-dimensional topology (see [3, 12]). Concrete calculus on the links $A_{m,n}$ and $B_{p,q}$ was important. The results made the contrast between $A_{m,n}$ (hyperbolic) and $B_{p,q}$ (Seifert) clear, which was one of the purpose of [KTYY] (see also [17]).

In this article, we focus another importance, the method itself to get some necessary conditions on the lens space surgery coefficients of a given link, by using Alexander polynomial and Reidemeister torsion. Our method satisfies that a result on a link L always extends to the links whose Alexander polynomials are same with that of L.

We will compare the Reidemeister torsion of the result M of Dehn surgery along a given link and that of a lens space $L(p,q)$ (in Example 3.4). Some necessary conditions are obtained from the value $\tau^{\psi_d}(M)$ of the Reidemeister torsion in the d-th cyclotomic field $\mathbb{Q}(\zeta_d)$ by *d-norm*, where $d(\geq 2)$ is a divisor of p. From the sequence of the equalities on $\tau^{\psi_d}(M)$s in $\mathbb{Q}(\zeta_d)$ for all divisors d of p (with a fixed combinatorial Euler structure of M), we take an *identity on symmetric Laurent polynomials*, as a lift of the equalities. We regard the identity as an equation of the surgery coefficient for M to be a lens space.

In the next section, we start with some definitions of the Reidemeister torsion. In Section 3, we review surgery formulae. In Section 4, we will study d-norms in the d-th cyclotomic field, and show a certain uniqueness of a symmetric polynomial as a lift of the sequence of the equalities in $\mathbb{Q}(\zeta_d)$s. In Section 5, we will explain the method to get some necessary condition of lens space surgery coefficients of a given link. In Section 6, as a demonstration, we will apply our method to Berge's link, which is one of the most famous targets in lens space surgery ([1]).

2. REIDEMEISTER TORSION

For a precise definition of the Reidemeister torsion, the reader refer to V. Turaev [14, 15]. Let X be a finite CW complex and $\pi : \tilde{X} \to X$ its maximal abelian covering. Then $\tilde{X}$ has a CW structure induced by that of X and π, and the cell chain complex $\mathbf{C}_*$ of $\tilde{X}$ has a

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E_L	the complement of L.
m_i, l_i	a meridian and a longitude of the i-th component K_i.
$[m_i], [l_i]$	their homology classes.
$\Delta_L(t_1, \dots, t_\mu)$	the Alexander polynomial of L, where t_i is represented by $[m_i]$.
$(L; r_1, \dots, r_\mu)$	the result of Dehn surgery along L, where $r_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ is the surgery coefficient of K_i.
V_i	the solid torus attached along K_i in the Dehn surgery.
m'_i, $[m'_i]$	a meridian of V_i, and its homology class.
l'_i, $[l'_i]$	an oriented core curve of V_i, and its homology class.

TABLE 1. Notations (for manifolds)

$\mathbb{Z}[H]$-module structure, where $H = H_1(X; \mathbb{Z})$ is the first homology of X. For an integral domain R and a ring homomorphism $\psi : \mathbb{Z}[H] \to R$, "the chain complex of $\tilde{X}$ related with ψ", denoted by $\mathbf{C}_*^\psi$, is constructed as $\mathbf{C}_* \otimes_{\mathbb{Z}[H]} Q(R)$, where $Q(R)$ is the quotient field of R. *The Reidemeister torsion of X related with ψ*, denoted by $\tau^\psi(X)$, is calculated from $\mathbf{C}_*^\psi$, and is an element of $Q(R)$ determined up to multiplication of $\pm\psi(h)$ $(h \in H)$. If $R = \mathbb{Z}[H]$ and ψ is the identity map, then we denote $\tau^\psi(X)$ by $\tau(X)$. We note that $\tau^\psi(X)$ is not zero if and only if $\mathbf{C}_*^\psi$ is acyclic.

Notation (for manifolds and homologies) Let $L = K_1 \cup \ldots \cup K_\mu$ be an oriented μ-component link in S^3. We will use the notations in Table 1.

Notation (for algebra) For a pair of elements A, B in $Q(R)$, if there exists an element $h \in H$ such that $A = \pm\psi(h)B$, then we denote the equality by $A \doteq B$. We will often take a field F and a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \to F$. We mainly use the d-th cyclotomic fields $\mathbb{Q}(\zeta_d)$ as F, where ζ_d is a primitive d-th root of unity.

3. SURGERY FORMULAE

Let E be a compact 3-manifold whose boundary ∂E consists of tori (E is possibly not E_L for a link L). We study the 3-manifold $M = E \cup V_1 \cup \cdots \cup V_n$ obtained by attaching solid tori V_is to E by attaching maps $f_i : \partial V_i \to \partial E$ ($\mathrm{Im}\,(f_i) \cap \mathrm{Im}\,(f_j) = \emptyset$ for $i \neq j$). By l'_i we denote the core of V_i. We let $\iota : E \hookrightarrow M$ denote the natural inclusion.

Lemma 3.1. (Surgery formula I) *If $\psi([l'_i]) \neq 1$ for every $i = 1, \dots, n$, then*

$$\tau^\psi(M) \doteq \tau^{\psi'}(E) \prod_{i=1}^{n} (\psi([l'_i]) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_*$ (ι_* is a ring homomorphism induced by ι).

For the case of the complement E_L of a μ-component link L in S^3 as in Table 1. The Reidemeister torsion is closely related with the Alexander polynomial.

Lemma 3.2. (Milnor [11]) *Let $\Delta_L(t_1, \dots, t_\mu)$ be the Alexander polynomial of a μ-component link $L = K_1 \cup \ldots \cup K_\mu$ in S^3, where a variable t_i is represented by the meridian* *of* K_i $(i = 1, \dots, \mu)$.

$$\tau(E_L) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & (\mu = 1), \\ \Delta_L(t_1, \dots, t_\mu) & (\mu \geq 2). \end{cases}$$

Next, we study the result of Dehn surgery $M = (L; p_1/q_1, \dots, p_\mu/q_\mu)$ along L. We take integers r_i and s_i satisfying $p_i s_i - q_i r_i = -1$.

Lemma 3.3. (Surgery formula II; T. Sakai [13], V. G. Turaev [14])

(1) *In the case* $M = (K; p/q)$ $(|p| \geq 2)$, *we have* $H = H_1(M) \cong \langle T \mid T^p = 1 \rangle \cong \mathbb{Z}/|p|\mathbb{Z}$, *where* T *is represented by the meridian* $[m]$. *For a divisor* d (≥ 2) *of* p, *we define a ring homomorphism* $\psi_d : \mathbb{Z}[H] \to \mathbb{Q}(\zeta_d)$ *by* $\psi_d(T) = \zeta_d$. *Then we have*

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod{p}$.

(2) *In the case* $M = (L; p_1/q_1, \dots, p_\mu/q_\mu)$ $(\mu \geq 2)$. *Let* F *be a field and* $\psi : \mathbb{Z}[H_1(M)] \to F$ *a ring homomorphism. If* $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ *for every* $i = 1, \dots, \mu$, *then we have*

$$\tau^{\psi}(M) \doteq \Delta_L(\psi([m_1]), \dots, \psi([m_\mu])) \prod_{i=1}^{\mu} (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

Example 3.4. The lens space $L(p, q)$ is obtained as $-p/q$-surgery along the unknot. By Lemma 3.3 (1), for a divisor $d \geq 2$ of p, we have

$$\tau^{\psi_d}(L(p, q)) \doteq (\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod{p}$.

4. Cyclotomic field and Polynomial

4.1. d-norm.

About algebraic fields, the reader refer to L. C. Washington [16] for example.

For an element x in the d-th cyclotomic field $\mathbb{Q}(\zeta_d)$, the *d-norm* of x is defined as

$$N_d(x) = \prod_{\sigma \in \mathrm{Gal}\,(\mathbb{Q}(\zeta_d)/\mathbb{Q})} \sigma(x),$$

where $\mathrm{Gal}\,(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ is the Galois group $(\cong (\mathbb{Z}/d\mathbb{Z})^\times)$ related with a Galois extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. The following is well-known.

Proposition 4.1.

(1) *If* $x \in \mathbb{Q}(\zeta_d)$, *then* $N_d(x) \in \mathbb{Q}$. *The map* $N_d : \mathbb{Q}(\zeta_d) \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ *is a group homomorphism.*

(2) *If* $x \in \mathbb{Z}[\zeta_d]$, *then* $N_d(x) \in \mathbb{Z}$.

By easy calculations, we have the following.

Lemma 4.2.

(1) $N_d(\pm\zeta_d) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3). \end{cases}$

(2) $N_d(1 - \zeta_d) = \begin{cases} \ell & (d \text{ is a power of a prime } \ell \geq 2), \\ 1 & (\text{otherwise}). \end{cases}$

About applications of d-norms, for example, see [5, 6, 7, 8, 9, 10].

4.2. **Reidemeister–Turaev torsion.**

Let M be a homology lens space with $H = H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then the Reidemeister torsion $\tau^{\psi_d}(M)$ of M related with ψ_d is determined up to multiplication of $\pm\zeta_d^m$ $(m \in \mathbb{Z})$, where $d \geq 2$ is a divisor of p and ψ_d is the same ring homomorphism as in Lemma 3.3 (1). Once we fix a basis of a cell chain complex for the maximal abelian covering of M as a $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}]/(t^p - 1)$-module, the value $\tau^{\psi_d}(M)$ is uniquely determined as an element of $\mathbb{Q}(\zeta_d)$ for every d. The choice of the basis up to "base change equivalence" is called a *combinatorial Euler structure* of M (cf. Turaev [15]). The Reidemeister torsion of a manifold with a fixed combinatorial Euler structure is said the *Reidemeister–Turaev torsion.*

We consider the sequence of the values $\tau^{\psi_d}(M)$ in $\mathbb{Q}(\zeta_d)$ of the Reidemeister–Turaev torsion for every divisor $d \geq 2$ of p, and regard them as a value sequence $\{\tau^{\psi_d}(M)\}_{d|p,d\geq 2}$ defined as below.

Definition 4.3. We define that a sequence of values $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ is a *value sequence (of degree p)* if $x_d \in \mathbb{Q}(\zeta_d)$ for every d. Two value sequences $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p,d\geq 2}$ are *equal* ($\mathbf{x} = \mathbf{y}$) if $x_d = y_d$ for every d. We are mainly concerned with the value sequence of type $\mathbf{x} = \{F(\zeta_d)\}_{d|p,d\geq 2}$ for a rational function $F(t) \in \mathbb{Q}(t)$. In such a case, we say that $\mathbf{x}$ is *induced by* $F(t)$ and that $F(t)$ is a *lift* of $\mathbf{x}$. A *control* of $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ by a trivial unit $u = \eta t^m \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ is defined by

$$u\mathbf{x} = \{\eta\zeta_d^m x_d\}_{d|p,d\geq 2},$$

where $\eta = 1$ or -1 (constant) and $m \in \mathbb{Z}$. Two value sequences $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p,d\geq 2}$ are *control equivalent* if there is a trivial unit $u \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ such that $\mathbf{y} = u\mathbf{x}$. A value sequence $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ is a *real value sequence* if x_d is a real number for every d.

Example 4.4. A value sequence $\mathbf{x}$ of degree 12 is in the form $\mathbf{x} = \{x_2, x_3, x_4, x_6, x_{12}\}$. The following two value sequences $\mathbf{x}, \mathbf{y}$ of degree 12 are not equal, but control equivalent for $u = t^6$.

$$\mathbf{x} = \{2, -1, -2, -1, 1\}, \qquad \mathbf{y} = \{2, -1, 2, 1, -1\}.$$

In fact, $\mathbf{x}$ and $\mathbf{y}$ is induced by $t^2 + t^{-2}$ and $t^4 + t^{-4}$, respectively.

Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then a sequence $\{\tau^{\psi_d}(M)\}_{d|p,d\geq 2}$ of the Reidemeister torsions of M with a combinatorial Euler structure is a value sequence of degree p. We say the value sequence a *torsion sequence* of M.

Lemma 4.5.

(1) *Let M and M' be homeomorphic homology lens spaces with $H_1(M) \cong H_1(M') \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then torsion sequences $\{\tau^{\psi_d}(M)\}_{d|p,d\geq 2}$ and $\{\tau^{\psi'_d}(M')\}_{d|p,d\geq 2}$ related with the corresponding ring homomorphisms ψ_d and ψ'_d (i.e., $\psi_d = \psi'_d \circ h_*$, where h_* is the induced homomorphism of the homeomorphism) are control equivalent.*

(2) *Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ $(p \geq 2)$. Then we can control a torsion sequence of M into a real value sequence.*

Proof. (1) It is easy to see.

(2) Here we let ζ denote any d-th primitive root (ζ_d) of unity. Since M is obtained by p/q-surgery along a knot K in a homology 3-sphere for some q (cf. [2]), and we can also

apply Lemma 2.5 (1) for the case, we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod p$. By the duality of the Alexander polynomial (cf. [11, 14, 15]), we may assume

$$\Delta_K(t) = \Delta_K(t^{-1}).$$

This is also a control of the combinatorial Euler structure of the exterior of K, which induces a control of a torsion sequence of M. We take an odd integer lift of $\bar{q}$. Then

$$\zeta^{\frac{1+\bar{q}}{2}} \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

is a real number for every d. □

Lemma 4.6. *If two real value sequences* $\mathbf{x} = \{x_d\}_{d|p,d\geq 2}$ *and* $\mathbf{y} = \{y_d\}_{d|p,d\geq 2}$ *of degree* p *are control equivalent satisfying* $\mathbf{y} = u\mathbf{x}$ *for a trivial unit* $u = \eta t^m \in \mathbb{Z}[t,t^{-1}]/(t^p - 1)$, *where* $\eta = \pm 1$ *and* $m \in \mathbb{Z}$, *then the possibility of* u *is restricted as follows:*

(i) *If* p *is odd, then* $u = 1$ *or* -1.
(ii) *If* p *is even, then* $u = 1,\ -1,\ t^{p/2}$ *or* $-t^{p/2}$.

Proof. Since the ratio $\zeta_p^m = \pm y_p/x_p$ is a real number, we have (i) $m \equiv 0 \pmod p$ if p is odd, and (ii) $m \equiv 0$ or $p/2 \pmod p$ if p is even. □

Definition 4.7. (Symmetric Laurent polynomial) A Laurent polynomial $F(t) \in \mathbb{Z}[t,t^{-1}]$ is *symmetric* if it is of the form

$$F(t) = a_0 + \sum_{i=1}^{\infty} a_i(t^i + t^{-i}),$$

where a_i is an integer for all $i = 1, 2, \ldots$ and $a_i = 0$ for every sufficiently large i. Note that, if $F(t)$ is a symmetric Laurent polynomial, the induced value sequence $\{F(\zeta_d)\}_{d|p,d\geq 2}$ is a real value sequence. We are concerned with symmetric Laurent polynomials that are lifts (in $\mathbb{Z}[t,t^{-1}]$) of a polynomial in the quotient ring $\mathbb{Z}[t,t^{-1}]/(t^p - 1)$. We say that $F(t)$ (as above) is *reduced* if $a_i = 0$ for all $i > [p/2]$. We often *reduce* the symmetric polynomials by using $t^i + t^{-i} = t^{p+i} + t^{-(p+i)}$ modulo $(t^p - 1)$. We let $\mathrm{red}(F(t))$ denote the reduction of $F(t)$ (i.e., $\mathrm{red}(F(t))$ is reduced and $\mathrm{red}(F(t)) = F(t)$ in $\mathbb{Z}[t,t^{-1}]/(t^p - 1)$). We will use a notation $\langle t^i \rangle = t^i + t^{-i}$, for short.

For a Laurent polynomial $F(t) \in \mathbb{Z}[t,t^{-1}]$, the *span* of $F(t)$ is the difference of the maximal degree of $F(t)$ and the minimal degree of $F(t)$, and we denote it by $\mathrm{span}\,(F(t))$.

Lemma 4.8. *Let* $N \geq 2$ *be an integer. Let* $F(t), G(t)$ *be symmetric Laurent polynomials and* $\mathbf{x} = \{F(\zeta_d)\}_{d|N,d\geq 2}$, $\mathbf{y} = \{G(\zeta_d)\}_{d|N,d\geq 2}$ *the induced real value sequences, respectively. If* $\mathbf{x}$ *and* $\mathbf{y}$ *are control equivalent, i.e.,* $u\mathbf{x} = \mathbf{y}$ *for a trivial unit* u *(here,* $u = 1$ *or* -1 *if* N *is odd,* $u = 1, -1, t^{N/2}$ *or* $-t^{N/2}$ *if* N *is even, by Lemma 4.6), and* $F(1) = G(1) = 0$, *then we have a congruence*

$$uF(t) \equiv G(t) \mod t^N - 1$$

Furthermore, assuming $\mathrm{span}\,(G(t)) \leq 2[N/2]$,

(i) *In the case that* $u = 1$ *or* -1 *and* $\mathrm{span}(F(t)) \leq N - 1$, *we have an identity* $uF(t) = G(t)$ *in* $\mathbb{Z}[t,t^{-1}]$.

(ii) *Otherwise (in the case that N is even and $u = \eta t^{N/2}$ with $\eta = 1$ or -1), we have $red(t^{N/2}F(t)) = \eta G(t)$ in $\mathbb{Z}[t, t^{-1}]$.*

Proof. By Chinese Remainder Theorem, we have a ring isomorphism:

$$\mathbb{Q}[t, t^{-1}]/\,(t^N - 1) \cong \bigoplus_{d|N, d\geq 1} \mathbb{Q}(\zeta_d),$$

where $f(t)$ in the left-hand side maps to the value sequences $\{f(\zeta_d)\}_{d|N,d\geq 2}$ in the right-hand side. The isomorphism implies the required congruence. □

Note that $F(t)$ and $t^{N/2}F(t)$ induce the control equivalent real value sequences by $u = t^{N/2}$, but $\mathrm{red}(t^{N/2}F(t)) \neq F(t)$ in general, see Example 4.4. Thus we have to care the case (ii) in the lemma. Here, we study relation between the coefficients of $F(t)$ and those of $\mathrm{red}(t^{N/2}F(t))$.

Lemma 4.9. *Let N be an even integer.*

$$\text{If } F(t) = a_0 + \sum_{i=1}^{N/2} a_i(t^i + t^{-i}), \text{ then } red(t^{N/2}F(t)) = b_0 + \sum_{i=1}^{N/2} b_i(t^i + t^{-i})$$

with

$$b_0 = 2a_{N/2},\ b_{N/2} = a_0/2 \text{ and } b_j = a_{N/2-j} \quad (j = 1, 2, \cdots, N/2 - 1).$$

Proof. It is because

$$t^{N/2}(t^j + t^{-j}) \;=\; t^{N/2+j} + t^{N/2-j} \;\equiv\; t^{(N/2-j)} + t^{-(N/2-j)} \quad \mod t^N - 1.$$

□

5. Method

Let $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be a link. We let M simply denote the result $(L; r_1, \ldots, r_\mu)$ of the Dehn surgery. We use the notations in Table 1.

Step 1 *Study the first homologies (the generators and relations), from the exterior E_L of L (Of course, $H_1(E_L;\mathbb{Z}) \cong \bigoplus_{i=1}^{\mu} \mathbb{Z}[m_i]$) to the result M, by attaching solid tori V_i one by one.*

The first (obvious) necessary condition for the result M of Dehn surgery to be a lens space $L(p, q)$ is

$$H_1(M;\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

Step 2 *Calculate the Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$ of L. Using Lemma 3.2 and Lemma 3.3, calculate the Reidemeister torsion $\tau^{\psi}(M)$ related with a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \to \mathbb{Q}(\zeta_d)$, where d (≥ 2) is a divisor of p.*

If M is homeomorphic to a lens space $L(p, q)$ (with undecided q), then their Reidemeister torsions are equal to each other. By Example 3.4, there exists integers i, j coprime to p with $0 < i, j < p$ (they are lifts of $(\mathbb{Z}/p\mathbb{Z})^{\times}/\{\pm 1\}$) such that

$$\tau^{\psi}(M) \doteq \frac{1}{(\zeta_d^i - 1)(\zeta_d^j - 1)} \quad \text{in } \mathbb{Q}(\zeta_d), \tag{1}$$

for each divisor d (≥ 2) of p. We can assume $i + j$ is even by retaking $p - j$ instead of j.

Step 3 *Using d-norm in $\mathbb{Q}(\zeta_d)$, studied in Subsection 4.1, to the equality (1), we have a necessary condition on the coefficient of lens space surgery.*

We fix a combinatorial Euler structure (multiple of trivial unit $\pm{\zeta_d}^k$), deform both hand-sides of the equality (1) into real values by Lemma 4.5(2). If M is homeomorphic to $L(p,q)$, we have a control equivalence between the real value sequence:

$$\{\tau^{\psi}(M)\}_{d|p,d\geq 2} \;=\; u\{\,\zeta_d^{\frac{i+j}{2}}(\zeta_d^i-1)^{-1}(\zeta_d^j-1)^{-1}\,\}_{d|p,d\geq 2},$$

where u is a trivial unit ± 1, or $\pm t^{p/2}$ (only in the case p is even). By Lemma 4.8, we have, via a congruence mod (t^p-1), an identity between symmetric Laurent polynomials. We regard the identity as an equation (on (i,j)) of the coefficients of lens space surgery.

Step 4 *By the equation, we have a necessary condition on the coefficient(s) of lens space surgery.*

6. Demonstration

We call the link in Figure 1 *Berge's link BL*. The compliment is a hyperbolic 3-manifold, known as Berge's manifold in [1]. The component K_1 is the famous pretzel knot $P(-2,3,7)$. The link, regarded as a knot in a solid torus (the exterior of the component K_2), admits two surgery coefficients yielding solid torus itself, and it is proved that such a hyperbolic link is unique [1]. We demonstrate our method in Section 5 to Berge's link,

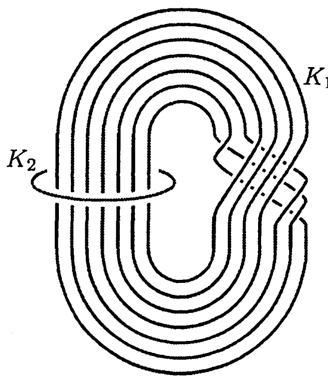

FIGURE 1. Berge link BL

to study lens space surgeries $M := (BL; r, 0)$, where $r=\alpha/\beta$ $(\alpha,\beta\in\mathbb{Z}, \gcd(\alpha,\beta)=1)$. We assume that $\beta\geq 1$.

(Step 1)

$$H_1(M)\cong\langle [m_1],[m_2]\mid [l_1]=[m_2]^7, [l_2]=[m_1]^7,\ [m_1]^{\alpha}[l_1]^{\beta}=1, [m_1]^7=1\rangle.$$

It is finite cyclic $\mathbb{Z}/p\mathbb{Z}$ if and only if $\gcd(\alpha,7)=1$, and then we have $p=7^2\beta=49\beta$. An element $T=[m_1]^{\gamma'}[m_2]^{\delta'}$ with $\alpha\delta'-7\beta\gamma'=-1$ is a generator: $T^{49\beta}=1$. We also have $[l_1']=[m_1]^{\gamma}[l_1]^{\delta}$ with $\alpha\delta-\beta\gamma=-1$, and

$$[m_1]=T^{7\beta},\ [m_2]=[l_2']=T^{-\alpha},\quad [l_1']=T^7.$$

(Step 2) The Alexander polynomial of Berge's link is

$$\Delta_{BL}(t,x) \doteq 1+t^3x+t^5x^2+t^8x^3+t^{11}x^4+t^{13}x^5+t^{16}x^6 = \sum_{i=0}^{6} t^{s_i}x^i,$$

where we define a sequence $(s_0, s_1, \cdots, s_6) = (0,3,5,8,11,13,16)$. This is not periodic, but we regard it as "Periodicity is broken a little". We let $M_1 = E_{BL} \cup V_1 = (BL; \alpha/\beta, -)$. We have, up to the ambiguity (multiplication $\pm T^k$),

$$\tau(M_1) \doteq \Delta_{BL}(T^{7\beta}, T^{-\alpha})(T^7-1)^{-1} = \left(\sum_{i=0}^{6} T^{7\beta s_i - \alpha i}\right)(T^7-1)^{-1}.$$

We take a divisor $d=7$ of $p=49\beta$ and let ζ denote a primitive 7-th root of unity. We use deformations

$$T^{7\beta s_i-\alpha i} = T^{-\alpha i}\left(T^{7\beta s_i}-1\right)+T^{-\alpha i}, \qquad \frac{T^{7\beta s_i}-1}{T^7-1} = 1+T^7+T^{14}+\cdots+T^{7(\beta s_i-1)}.$$

For a ring homomorphism ψ satisfying $\psi(T)=\zeta$ with $\xi=\zeta^{-\alpha}$ (then ξ is still a primitive unity, since $\gcd(\alpha,7)=1$),

$$\begin{aligned}\tau^{\psi}(M) &\doteq \left\{\beta(\zeta^{-\alpha}-1)\left(\sum_{i=0}^{6} s_i\zeta^{-\alpha i}\right)-\alpha\right\}(\zeta^{-\alpha}-1)^{-2}\\ &= \left\{\beta(\xi-1)\left(\sum_{i=0}^{6} s_i\xi^{i}\right)-\alpha\right\}(\xi-1)^{-2}.\end{aligned}$$

In the 7-th cyclotomic field $\mathbb{Q}(\zeta_7)$, using the equalities $\xi^7=1$ and $1+\xi+\xi^2+\xi^3+\xi^4+\xi^5+\xi^6=0$,

$$\begin{aligned}(\xi-1)\sum_{i=0}^{6} s_i\xi^i &= -3\xi-2\xi^2-3\xi^3-3\xi^4-2\xi^5-3\xi^6+16\\ &= -3\xi-2\xi^2-3\xi^3-3\xi^4-2\xi^5-3\xi^6+16\\ &\quad +3(1+\xi+\xi^2+\xi^3+\xi^4+\xi^5+\xi^6)\\ &= 19+\xi^2+\xi^5\\ &= 19+\xi^2+\xi^{-2}.\end{aligned}$$

The Reidemeister–Turaev torsion of Dehn surgery $M=(BL;\alpha/\beta,0)$ is

(2) $$\tau^{\psi}(M) \doteq \{\beta(\xi^2+\xi^{-2})-(\alpha-19\beta)\}(\xi-1)^{-2}.$$

Now, suppose that M is a lens space $L(p,q)$ with $p=49\beta$ (by Step 1) and undecided q. Then there exist integers i,j coprime to p with $0<i,j<p$ such that

(3) $$\tau^{\psi}(M) \doteq (\xi^i-1)^{-1}(\xi^j-1)^{-1}.$$

We can assume $i+j$ is even. We treat with i,j mod 7 ($i,j\in\{1,2,3,4,5,6\}$), since $d=7$.

(Step 3) Using Lemma 4.2 on d-norm with $d=7$ on (2) and (3), we have a necessary condition for the Dehn surgery $M=(BL;\alpha/\beta,0)$ to be a lens space:

$$N_d\left(\beta(\xi^2+\xi^{-2})-(\alpha-19\beta)\right)=1.$$

Roughly, it means $r=\alpha/\beta$ is near 19.

(Step 4) We set $\alpha' = \alpha - 19\beta$. By (2) and (3), we have

$$\xi\left\{\beta(\xi^2+\xi^{-2})-\alpha'\right\}(\xi-1)^{-2} = \pm\xi^{(i+j)/2}(\xi^i-1)^{-1}(\xi^j-1)^{-1}.$$

We regard it as an equality between real value sequence. Without loss of generality, we assume $0 < i < d/2$ (i.e., $i = 1, 2$ or 3), $i \leq j$, and define $f = (i+j)/2, e = (j-i)/2$. The equality lifts as an identity of symmetric Laurent polynomial

$$(4) \qquad (\beta\langle t^2\rangle - \alpha')(\langle t^f\rangle - \langle t^e\rangle) = \pm(\langle t\rangle - 2),$$

in $\mathbb{Z}[t,t^{-1}]/(t^7-1)$, where $\langle t^i\rangle = t^i + t^{-i}$, as in Definition 4.7. The left-hand side $F(t)$ is expanded to

$$\beta\langle t^{f+2}\rangle + \beta\langle t^{f-2}\rangle - \alpha'\langle t^f\rangle - \beta\langle t^{e+2}\rangle - \beta\langle t^{e-2}\rangle + \alpha'\langle t^e\rangle.$$

We regard the identity (4) as an equation on (f,e): It is a necessary condition on (α',β) for the equation to have a solution (f,e). Since $f \neq e$ is obvious and $\langle t^4\rangle = \langle t^3\rangle$, $\langle t^5\rangle = \langle t^2\rangle$ mod (t^7-1), we only have to consider six cases

$$(f,e) = (1,0),\ (2,0),\ (3,0),\ (2,1),\ (3,1),\ (3,2).$$

Note that $\langle t^{-x}\rangle = \langle t^x\rangle$ and $\langle t^0\rangle = 2$.

(f,e)	$F(t)$	(α',β)
$(1,0)$	$\beta\langle t^3\rangle - 2\beta\langle t^2\rangle - (\alpha'-\beta)\langle t^1\rangle + 2\alpha'$	No
$(2,0)$	$\beta\langle t^3\rangle - (\alpha'+2\beta)\langle t^2\rangle + 2(\alpha'+\beta)$	No
$(3,0)$	$-\alpha'\langle t^3\rangle - \beta\langle t^2\rangle + \beta\langle t^1\rangle + 2\alpha'$	No
$(2,1)$	$-\alpha'\langle t^2\rangle + (\alpha'-\beta)\langle t^1\rangle + 2\beta$	$(\alpha',\beta) = (0,1)$
$(3,1)$	$-(\alpha'+\beta)\langle t^3\rangle + \beta\langle t^2\rangle + \alpha'\langle t^1\rangle$	No
$(3,2)$	$-(\alpha'+\beta)\langle t^3\rangle + (\alpha'+\beta)\langle t^2\rangle + \beta\langle t^1\rangle - 2\beta$	$(\alpha',\beta) = (-1,1)$

Since $\alpha' = \alpha - 19\beta$, $(\alpha',\beta) = (0,1)$ (and $(-1,1)$, respectively) corresponds to $\alpha/\beta = 19$ (and 18). We have the required conclusion (pointed out in [1]):

Berge's link BL yields a lens space as $(BL; r, 0)$ only if $r = 19$ or $r = 18$.

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